ON THE APPLICATION OF SELF-SIMILAR SOLUTIONS TO DYNAMIC PROBLEMS IN A PLASTIC MEDIUM

(O PRIMENENII AVTOMODEL'NYKH RESHENII K ZADACHAM DINAMIKI PLASTICHESKOI SREDY)

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Self-similar techniques allow one to find exact solutions in a number of problems, while in a wider class of problems they allow one to obtain approximate solutions. These techniques are examined in general terms in [1]. There exists also a large number of papers, in particular the work of Barenblatt [2,3], Barenblatt and Zel'dovich [4,5], in which the method of self-similar solutions is successfully applied to various problems of the nonstationary motion of continuous media.

Self-similar solutions as a means of obtaining exact solutions of the equations of dynamic plasticity in plane problems have been investigated by the author [6]. The aim of the present note is to show how approximate solutions for a wider class of problems may be obtained.

1. The nonsteady motion of an ideally plastic medium in a state of plane strain is described by the following system of equations [7]:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} - \rho \frac{\partial u}{\partial t} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma}{\partial y} - \rho \frac{\partial v}{\partial t} = 0$$

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2$$

$$\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\partial v / \partial x + \partial u / \partial y}{\partial u / \partial x - \partial v / \partial y}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Expressing the stresses in terms of the functions χ and ϕ by means of the formulas

$$\left.\begin{matrix}\sigma_x \\ \sigma_y\end{matrix}\right\} = k\,(2\chi \pm \cos 2\varphi), \qquad \tau_{xy} = k \sin 2\varphi$$

we obtain a system of the form

$$\frac{\partial \chi}{\partial x} - \sin 2\varphi \frac{\partial \varphi}{\partial x} + \cos 2\varphi \frac{\partial \varphi}{\partial y} - \frac{\rho}{2k} \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial\chi}{\partial y}+\cos 2\varphi\frac{\partial\varphi}{\partial x}+\sin 2\varphi\frac{\partial\varphi}{\partial y}-\frac{\rho}{2k}\frac{\partial v}{\partial t}=0 \tag{1}$$

$$2\frac{\partial u}{\partial x}-\cot 2\varphi\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=0,\qquad 2\frac{\partial v}{\partial y}+\cot 2\varphi\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=0$$

If we introduce new independent variables

$$\lambda=\frac{x}{at},\qquad \mu=\frac{y}{at},\qquad \tau=\frac{t}{t_0},\qquad a=\frac{2k}{\rho} \tag{2}$$

then the system (1) takes on the form

$$\begin{aligned}
&\frac{\partial\chi}{\partial\lambda}-\sin 2\varphi\frac{\partial\varphi}{\partial\lambda}+\cos 2\varphi\frac{\partial\varphi}{\partial\mu}+\lambda\frac{\partial u}{\partial\lambda}+\mu\frac{\partial u}{\partial\mu}-\tau\frac{\partial u}{\partial\tau}=0\\
&\frac{\partial\chi}{\partial\mu}+\cos 2\varphi\frac{\partial\varphi}{\partial\lambda}+\sin 2\varphi\frac{\partial\varphi}{\partial\mu}+\lambda\frac{\partial v}{\partial\lambda}+\mu\frac{\partial v}{\partial\mu}-\tau\frac{\partial v}{\partial\tau}=0\\
&2\frac{\partial u}{\partial\lambda}-\cot 2\varphi\left(\frac{\partial v}{\partial\lambda}+\frac{\partial u}{\partial\mu}\right)=0,\qquad 2\frac{\partial v}{\partial\mu}+\cot 2\varphi\left(\frac{\partial v}{\partial\lambda}+\frac{\partial u}{\partial\mu}\right)=0
\end{aligned} \tag{3}$$

For $t_0\to\infty$ this system passes to the system of equations of self-similar motions which was examined in [6]. The quantity t_0 can be regarded as a characteristic time of the nonstationary process, which for self-similar motions is infinite, but which in a wider class of motions is finite. To obtain an approximate solution for large values of t_0 one naturally expands in powers of τ

$$\begin{aligned}
\chi&=\chi^{(0)}+\tau\chi^{(1)}+\ldots, & u&=u^{(0)}+\tau u^{(1)}+\ldots\\
\varphi&=\varphi^{(0)}+\tau\varphi^{(1)}+\ldots, & v&=v^{(0)}+\tau v^{(1)}+\ldots
\end{aligned} \tag{4}$$

Substituting the expansions (1) into Equations (3), and setting equal to zero the coefficients of like powers of τ, we obtain successive systems of equations of zeroth, first, second, etc. order of approximation.

The zeroth-order system is a system of self-similar equations. Let us examine in detail the first-order approximation.

The corresponding equations have the form

$$\begin{aligned}
&\frac{\partial\chi^{(1)}}{\partial\lambda}-\sin 2\varphi^{(0)}\frac{\partial\varphi^{(1)}}{\partial\lambda}+\cos 2\varphi^{(0)}\frac{\partial\varphi^{(1)}}{\partial\mu}+\lambda\frac{\partial u^{(1)}}{\partial\lambda}+\mu\frac{\partial u^{(1)}}{\partial\mu}-\\
&\qquad -2\varphi^{(1)}\left(\frac{\partial\varphi^{(0)}}{\partial\lambda}\cos 2\varphi^{(0)}-\frac{\partial\varphi^{(0)}}{\partial\mu}\sin 2\varphi^{(0)}\right)-u^{(0)}=0\\
&\frac{\partial\chi^{(1)}}{\partial\mu}+\cos 2\varphi^{(0)}\frac{\partial\varphi^{(1)}}{\partial\lambda}+\sin 2\varphi^{(0)}\frac{\partial\varphi^{(1)}}{\partial\mu}+\lambda\frac{\partial v^{(1)}}{\partial\lambda}+\mu\frac{\partial v^{(1)}}{\partial\mu}-\\
&\qquad -2\varphi^{(1)}\left(\frac{\partial\varphi^{(0)}}{\partial\lambda}\sin 2\varphi^{(0)}-\frac{\partial\varphi^{(0)}}{\partial\mu}\cos 2\varphi^{(0)}\right)-v^{(0)}=0
\end{aligned} \tag{3}$$

$$2\frac{\partial u^{(1)}}{\partial\lambda}-\cot 2\varphi^{(0)}\left(\frac{\partial v^{(1)}}{\partial\lambda}+\frac{\partial u^{(1)}}{\partial\mu}\right)+2\varphi^{(1)}\left(2\frac{\partial u^{(0)}}{\partial\lambda}\cot 2\varphi^{(0)}+\frac{\partial v^{(0)}}{\partial\lambda}+\frac{\partial u^{(0)}}{\partial\mu}\right)=0$$

$$2\frac{\partial v^{(1)}}{\partial\mu}+\cot 2\varphi^{(0)}\left(\frac{\partial v^{(1)}}{\partial\lambda}+\frac{\partial u^{(1)}}{\partial\mu}\right)+2\varphi^{(1)}\left(2\frac{\partial v^{(0)}}{\partial\mu}\cot 2\varphi^{(0)}-\frac{\partial v^{(0)}}{\partial\lambda}-\frac{\partial u^{(0)}}{\partial\mu}\right)=0$$

This is a system of four linear equations in the unknowns $\chi^{(1)}$, $\phi^{(1)}$, $u^{(1)}$, $v^{(1)}$. We assume that the solution of the self-similar equations has already been obtained; hence, all quantities of zeroth order can be regarded as known. We then determine the characteristics of the system (5). To do this it is necessary to solve the equations of the system for the partial derivatives, together with the expressions for the total differentials of the functions $\chi^{(1)}$, $\phi^{(1)}$, $u^{(1)}$, $v^{(1)}$. The equation for the determination of the directions of the characteristics has the form $|a_{ik}| = 0$, where the coefficients of the determinant a_{ik} are the following:

$$a_{11}=1,\quad a_{12}=-\sin 2\varphi^{(0)}-\cos 2\varphi^{(0)}\frac{d\lambda}{d\mu},\quad a_{13}=\lambda+\mu\frac{d\lambda}{d\mu},\quad a_{14}=0$$

$$a_{21}=\frac{d\lambda}{d\mu},\quad a_{22}=\cos 2\varphi^{(0)}-\sin 2\varphi^{(0)}\frac{d\lambda}{d\mu},\quad a_{23}=0,\quad a_{24}=\lambda+\mu\frac{d\lambda}{d\mu}$$

$$a_{31}=0,\quad a_{32}=0,\quad a_{33}=2+\cot 2\varphi^{(0)}\frac{d\lambda}{d\mu},\quad a_{34}=-\cot 2\varphi^{(0)}$$

$$a_{41}=0,\quad a_{42}=0,\quad a_{43}=-\cot 2\varphi^{(0)}\frac{d\lambda}{d\mu},\quad a_{44}=\cot 2\varphi^{(0)}-2\frac{d\lambda}{d\mu}$$

It is easily seen that the fourth-order determinant decomposes into two determinants of second order, giving two equations for the determination of $d\mu/d\lambda$. It turns out that each of these equations leads to an expression for the characteristic directions of the following form:

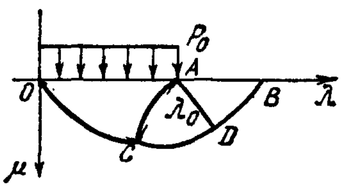
$$\frac{d\mu}{d\lambda}=\tan\left(\varphi^{(0)}\pm\frac{\pi}{4}\right) \tag{7}$$

Hence, at each point of the λ, μ-plane there are two real characteristic directions, each with a multiplicity of two, so the system (5) is hyperbolic in the entire λ, μ-plane.

The hyperbolicity of the system (5) allows one to develop numerical methods of solution that are analogous to the methods applied in static plasticity and gasdynamics. One of the possible versions of difference equations, which allow one to carry out the numerical integration of the system of nonlinear equations of self-similar motion, was investigated in [6]. In the present case the difference equations will be somewhat more complicated because the unknown function $\phi^{(1)}$ appears along with the velocities in the last two equations for the first-order approximation. On the other hand, the present problem is simpler in that the system (5) is linear, so that its characteristics are known beforehand. It is essential to note that, as follows from Formula (7), the characteristic directions of system (5) coincide with the characteristic directions of the system of self-similar equations, so that they may be considered as known if the zeroth-order approximation is known.

2. The equations which have been obtained allow one to find approximate solutions for a number of dynamic plasticity problems. The self-similar problem of the propagation of a constant pressure p over the surface of a half-space was investigated in [6]. Let us assume that it is required to solve the more difficult problem where p depends on time. We represent the pressure on the surface of the body in the form

$$p = p_0 f(\lambda), \qquad f(\lambda) = \begin{cases} 1 & (\lambda < \lambda_0) \\ 0 & (\lambda > \lambda_0) \end{cases}$$



If p is expanded in a power series

$$p_0 = p_0^{(0)} + \tau p_0^{(1)} + \ldots$$

then for the equations of the first-order approximation we have the following combination of boundary-value problems (figure). In the triangle *ABD* we have a Cauchy problem; in the fan *ADC* a degenerate Goursat problem; and in *OAC* a mixed problem. As a result, one may find the stresses and velocities in the entire plastically deformed region of the λ, μ-plane. In a completely analogous fashion one may solve the problem of the propagation of pressure when the wave-front velocity depends on time.

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